Chaos in monopole sector of the Georgi - Glashow model.

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Abstract

A spherically symmetric excitations of the Polyakov - 't Hooft monopole are considered. In the framework of the geodesics deviation equation it is found that in the large mass Higgs sector a signature of chaos occurs.

1 Introduction

Because of extreme complication of the evolution of nonlinear dynamical systems a great career make methods originated in Poincare section method [1]. There is also growing pressure on evaluating Lapunov exponents and Hausdorff dimension which are the most restrictive (global) characteristics of the behavior of the dynamical system [2]. Simultaneously there is also a tendency to improve and find new local criterions of chaos which are much simpler in application than the global one. Historically the first criterion was based on the fact that in the neighbourhood of the fixed points the dynamics is dominated by the linear terms [3]. This method in more formal way could be expressed as a condition (Toda criterion) on the Gauss curvature of the potential energy surface [4]. The criterion based on the geodesics deviation equation in space equipped in Jacobi metric [5] seems to be the best geometrically motivated one.

On the other hand in particle physics [6], cosmology [7] and condensed matter physics [8] there is (caused by growing number of applications) tremendous interest in dynamics of the Yang - Mills fields. On classical level, studies of this subject concentrate mainly on seeking for new solutions and their excitations [9].

In this paper extending approach of the paper [10] we apply local geometric criterion of chaos to monopole sector of the Georgi - Glashow model. Using the local criterion of chaos we investigate the neighbourhood of the core of the Polyakov - 't Hooft monopole. We show that in the large mass Higgs sector of the model, excitations of the monopole behaves in chaotic way. On the other hand for sufficiently light Higgs fields trajectories do not diverge.

In the next section we will remind basic facts about Georgi - Glashow model and monopole solutions. Then we will use the criterion of sensitive dependence on initial conditions based on geodesic deviation equation. The suitable form of this criterion is prepared in appendix. As a result we shall discover difference of the dynamics of the light and heavy Higgs mass sectors. Last section contains some remarks.

2 Spherical excitations of the core of the Polyakov - Thooft monopole.

The Georgi - Glashow model is $SO(3)$ gauge invariant system consisting of a Higgs multiplet ϕ^a $(a = 1, 2, 3)$ transforming as a vector in the adjoint representation of the gauge group and the gauge fields $A_\mu = A^a{}_\mu T^a$.

In the same representation of the gauge group generators take the form $(T^a)_{bc} = -if^a{}_{bc}$, where $f^{abc} = \epsilon^{abc}$. They also satisfy the relation $[T^a, T^b] = if^{abc}T^c$. The lagrangian of the model is given by

$$\mathcal{L} = -\frac{1}{4}F^a{}_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi), \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \tag{2}$$

are the field strengths,

$$D_\mu\phi^a = \partial_\mu\phi^a - e\epsilon^{abc}A^b{}_\mu\phi^c \tag{3}$$

are covariant derivatives of the Higgs fields and

$$V(\phi) = \frac{1}{4}\lambda(\phi^2 - v^2)^2 \tag{4}$$

is a Higgs potential. The Euler - Lagrange equations for lagrangian (1)

$$(D_\nu F^{\mu\nu})^a = -e\epsilon^{abc}\phi^b(D^\mu\phi)^c, \tag{5}$$

$$D_\mu D^\mu\phi^a = -\lambda\phi^a(\phi^2 - v^2). \tag{6}$$

The field strengths also satisfy the Bianchi identities. Equations of motion have finite energy and time independent solutions. Existence and stability of these solutions is a result of nontrivial structure of the Higgs vacuum $\mathcal{N} = \{\phi^a : \phi^a\phi^a = v^2\}$ which has topology of a two sphere. The asymptotic Higgs fields $\phi_{r\to\infty}$ which are mappings from spatial infinity to the vacuum manifold of the Higgs fields could be divided into infinite number of inequivalent homotopy classes $\Pi_2(\mathcal{N}) = Z$. The transition (by continuos deformation) from one to another class needs infinite amount of the energy what makes these solutions stable.

A particular example of the monopole is given by Polyakov - 't Hooft (P-T) ansatz

$$\phi^a = x^a H(r) \equiv \Phi^a, \quad {A_k}^a = \epsilon_{aki} x^i K(r) \equiv {W_k}^a, \quad {A_0}^a = 0, \tag{7}$$

where H and K have the following asymptotic behaviour $H \stackrel{r\to 0}{\longrightarrow} H_* = const$, $K \stackrel{r\to 0}{\longrightarrow} K_* = const$ and $H \stackrel{r\to\infty}{\longrightarrow} \frac{1}{r}v$, $K \stackrel{r\to\infty}{\longrightarrow} -\frac{1}{e}\frac{1}{r^2}$. Monopole of this form carries one unit of topological charge.

From now we shall confine ourselves to $n = 1$ monopole sector of the Georgi - Glashow model.

Following the approach of paper [10] we consider time deformations of the P-T ansatz

$$\phi^a = \Phi^a f(t), \quad {A_k}^a = {W_k}^a g(t), \quad {A_0}^a = 0, \tag{8}$$

where $f > 0$ and $g > 0$ are unknown functions of time.

Lagrangian density on this new ansatz takes the form

$$\mathcal{L} = \frac{1}{2}\left(\Phi^a\Phi^a \dot{f}^2 + {W_k}^a {W_k}^a \dot{g}^2\right) - V(f,g) \tag{9}$$

where

$$\begin{aligned} V \;=\; & \frac{1}{4}\lambda(\Phi^a\Phi^a f^2 - v^2)^2 + \frac{1}{4}{F_{sk}}^a {F_{sk}}^a g^2 - \frac{1}{2}e\epsilon^{abc}{F_{sk}}^a {W_s}^b {W_k}^c g^3 \\ & + \frac{1}{4}e^2[({W_s}^a {W_s}^a)({W_k}^b {W_k}^b) - ({W_s}^a {W_k}^a)({W_s}^b {W_k}^b)]g^4 \\ & + \frac{1}{2}(\partial_k\Phi^a)(\partial_k\Phi^a)f^2 - e\epsilon^{abc}(\partial_k\Phi^a){W_k}^b\Phi^c g f^2 \\ & + \frac{1}{2}e^2[({W_k}^a {W_k}^a)(\Phi^b\Phi^b) - ({W_k}^a\Phi^a)({W_k}^b\Phi^b)]g^2 f^2 \end{aligned} \tag{10}$$

As the homogenous time dependent shift of the fields (7) in the whole space costs infinite amount of the energy it is unphysical. Therefore we confine our interest only to excitations of the core of the P-T monopole i.e. located in region surrounding zero of the Higgs fields.

We implement locality of these excitations by taking zero ($r \to 0$) asymptotics of the shape functions H and K

$$\mathcal{L} = \frac{1}{2}\left(r^2 {H_*}^2 \dot{f}^2 + 2r^2 {K_*}^2 \dot{g}^2\right) - V(f,g), \tag{11}$$

where

$$V = 6{K_*}^2g^2 + 2er^2{K_*}^3g^3 + \frac{1}{2}e^2r^4{K_*}^4g^4 + \frac{3}{2}{H_*}^2f^2 + \\ + 2er^2{H_*}^2K_*f^2g + e^2r^4{K_*}^2{H_*}^2f^2g^2 + \frac{1}{4}\lambda(r^2{H_*}^2f^2 - v^2)^2. \quad (12)$$

The effective lagrangian of the spherical excitations is in natural way defined by integration of the lagrangian density over a sphere of radius r_0 which contains the whole excitation $L = \int_{S_{r_0}} d^3x\mathcal{L} = 4\pi\int_0^{r_0} drr^2\mathcal{L}$. Local (regular in time) excitations have been found for large amount of solitonic solutions [9]. We attempt to check possibility of existence not only regular but also chaotic solutions. After rescaling by the overall factor $\frac{4\pi}{5}{r_0}^5$ the monopole effective lagrangian $q^1 = f$, $q^2 = g$ has the form

$$L = \frac{1}{2}\left[{H_*}^2(\dot{q}^1)^2 + 2{K_*}^2(\dot{q}^2)^2\right] - V(q^1, q^2), \quad (13)$$

$$V = \frac{10}{{r_0}^2}{K_*}^2(q^2)^2 + 2e{K_*}^3(q^2)^3 + \frac{5}{14}e^2{r_0}^2{K_*}^4(q^2)^4 + \frac{5}{2{r_0}^2}{H_*}^2(q^1)^2 + \\ + 2e{H_*}^2K_*(q^1)^2(q^2) + \frac{5}{7}e^2{r_0}^2{K_*}^2{H_*}^2(q^1)^2(q^2)^2 + \\ + \frac{1}{4}\lambda\left(\frac{5}{7}{r_0}^2{H_*}^4(q^1)^4 - 2v^2{H_*}^2(q^1)^2 + \frac{5}{3{r_0}^2}v^4\right) \quad (14)$$

Now we shall use, based on the geodesic deviation equation, local criterion of chaos. In two dimensions system allows chaotic behaviour if the Gauss curvature of the geometry determined by corresponding Jacobi metric is negative. If the Gauss curvature is positive and admissible for the physical trajectories region has no boundary then chaos does not occur. According to considerations of appendix (A8) the extrinsic curvature is

$$\hat{K} = \frac{1}{4(E-V)^3}\left\{(E-V)\left[\frac{1}{{H_*}^2}\frac{\partial^2 V}{\partial(q^1)^2} + \frac{1}{2{K_*}^2}\frac{\partial^2 V}{\partial(q^2)^2}\right] + \left[\frac{1}{{H_*}^2}\left(\frac{\partial V}{\partial q^1}\right)^2 + \frac{1}{2{K_*}^2}\left(\frac{\partial V}{\partial q^2}\right)^2\right]\right\} \quad (15)$$

In admissible for trajectories region of the configuration space $(E-V) > 0$. The second square bracket is also positive. Explicit differentiation of the potential (14) shows that $\frac{\partial^2 V}{\partial(q^2)^2} > 0$ and $\frac{\partial^2 V}{\partial(q^1)^2} = -\frac{1}{2}M_H + P(q^1, q^2)$, where $M_H =$

$2\lambda v^2$ is a Higgs mass and $P = \frac{5}{{r_0}^2}{H_*}^2 + 4e{H_*}^2K_*(q^2) + \frac{10}{7}e^2{r_0}^2{K_*}^2{H_*}^2(q^2)^2 + \frac{15}{7}\lambda{r_0}^2{H_*}^4(q^1)^2 > 0$ is positive function. From equation (15) it follows that the sign of the Gauss curvature is controlled by the value of the Higgs mass. For small Higgs masses $\hat{K}$ is positive and excitations evolve in regular way.

If mass of Higgs is sufficiently large then $\hat{K}$ became negative and chaos may occur.

3 Remarks

We considered spherically symmetric time - dependent deformations of the P-T monopole. This deformation could be obtained by the choice of the initial configuration which is deformed in the center and pure P-T monopole (7) at spatial infinity. In the heavy Higgs sector the local GDE criterion of chaos reveals chaotic behaviour of the fields. Because of the spherical symmetry we do not expect any radiation in this system which convince us that the excitation does not disappear.

4 Appendix

Our purpose is to calculate the Gauss curvature $\hat{K}$ of the surface described by the Jacobi metric. Let us start from evaluating the components of the Riemann tensor. The most efficient way of evaluating the components of the curvature tensor has its origin in Cartan structural equations

$$d\Theta^a + {\omega^a}_b \wedge \Theta^b = 0, \tag{A.1}$$

$$\hat{R}^a_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b, \tag{A.2}$$

where Θ^a is the orthogonal co-base, i.e. such that $\hat{g} = \Theta^1 \otimes \Theta^1 + ... + \Theta^N \otimes \Theta^N$. The first equation ensures the torsionless of the connection and the second one is a definition of the curvature 2 - form.

Let us confine to the two dimensional manifold equipped in metric

$$\hat{g} = \alpha dx \otimes dx + \beta dy \otimes dy, \tag{A.3}$$

where $\alpha = \alpha(q^1, q^2)$, $\beta = \beta(q^1, q^2)$.

In orthonormal co-base $\Theta^1 = \sqrt{\alpha} dq^1$, $\Theta^2 = \sqrt{\beta} dq^2$ metric (A3) can be expressed in standard form

$$\hat{g} = \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2. \tag{A.4}$$

In two dimensions the definition of the curvature 2-form (A2) is even simpler

$$\hat{R}^a_b = d\omega^a{}_b. \tag{A.5}$$

Evaluation of the curvature is extremely simple because the connection 1-form has the only one independent component (which follows from equation $d\hat{g} = 0$) $\omega^1{}_2 = -\omega^2{}_1 = \omega_{12} = -\omega_{21}$. The first Cartan structural equation leads to

$$\omega^1{}_2 = \frac{1}{2\sqrt{\alpha\beta}} \left(\frac{1}{\sqrt{\alpha}} \frac{\partial \alpha}{\partial q^2} \Theta^1 - \frac{1}{\sqrt{\beta}} \frac{\partial \beta}{\partial q^1} \Theta^2 \right). \tag{A.6}$$

On the other hand we use the fact that $\hat{R}^1_2$ component of the Riemann curvature tensor can be expressed by Gauss curvature

$$d\omega^1{}_2 = \hat{R}^1_2 = \hat{K} \Theta^1 \wedge \Theta^2. \tag{A.7}$$

Now the Gauss curvature computation is a straightforward substitution of $\omega^1{}_2$ into equation (A7)

$$\hat{K} = -\frac{1}{2\alpha\beta} \left(\frac{\partial^2 \alpha}{\partial (q^2)^2} + \frac{\partial^2 \beta}{\partial (q^1)^2} \right) + \frac{1}{4\alpha^2\beta^2} \left[\left(\frac{\partial \alpha}{\partial q^2} \right) \frac{\partial}{\partial q^2} (\alpha\beta) + \left(\frac{\partial \beta}{\partial q^1} \right) \frac{\partial}{\partial q^1} (\alpha\beta) \right]. \tag{A.8}$$

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